$$\text{Mass } \underset{[kg]}{M} = \iiint_R \rho(x,y,z) \cdot dV$$

density $[kg/m^3] \cdot [m^3]$

Center of mass

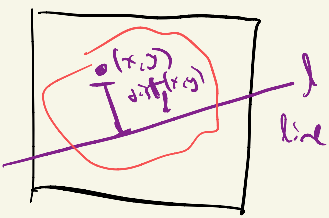
$$\bar{x} = \frac{1}{M} \iiint_R x\,\rho(x,y,z)\,dV$$

⋮

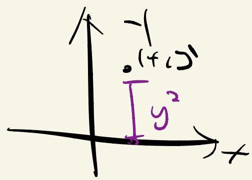
2nd moment: pick axis of rotation. Then just weight

$\mathbb{R}^2$:

$$I_\ell = \iint_R (\text{dist}_\ell(x,y))^2\,\sigma(x,y)$$

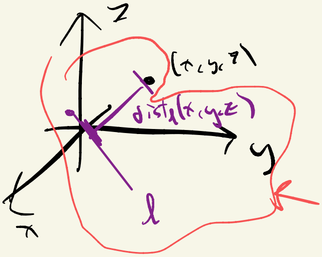


(ex: $I_x = \iint_R y^2\,\sigma(x,y)$



$\mathbb{R}^3$:

$$I_\ell = \iiint_R (\text{dist}_\ell(x,y,z))^2\,\rho(x,y,z)\,dV$$

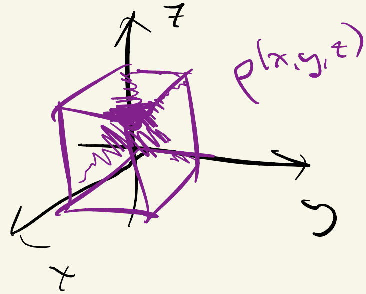


Think about flipping and angular momentum!

Ex:

A cube in the 1st octant bounded by $x=1$, $y=1$, $z=1$

$$\rho(x,y,z) = x+y+z+1$$



Mass:

$$\text{Mass} = \iiint_R \rho(x,y,z)\,dV$$

$$= \int_0^1\int_0^1\int_0^1 x+y+z+1 \, dz\,dy\,dx$$

$$= \int_0^1\int_0^1 \left[xz + yz + \frac{1}{2}z^2 + z\right]_{z=0}^{z=1} dy\,dx$$

$$= \int_0^1\int_0^1 x + y + \frac{3}{2} \, dy\,dx$$

$$= \int_0^1 \left[xy + \frac{1}{2}y^2 + \frac{3}{2}y\right]_{y=0}^{y=1} dx$$

$$= \int_0^1 x + \frac{4}{2} \, dx$$

$$= \left[\frac{1}{2}x^2 + \frac{4}{2}x\right]_{x=0}^{x=1}$$

$$= \frac{5}{2}$$

Center of mass:

$$\bar{x} = \frac{1}{M}\iiint_R x\,\rho(x,y,z)\,dV$$

$$= \frac{2}{5}\int_0^1\int_0^1\int_0^1 x^2+yx+zx+x\,dz\,dy\,dx$$

$$= \vdots$$

$$= \frac{8}{15}$$

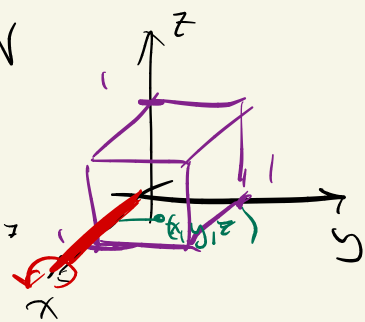
Symmetry! $\rho(x,y,z) = x+y+z$

Same integral for $\bar{y}$ and $\bar{z}$

$$\Rightarrow (\bar{x},\bar{y},\bar{z}) = \left(\frac{8}{15},\frac{8}{15},\frac{8}{15}\right)$$

Moment:

$$I_x = \iiint_R \left(\text{dist}_{(x\text{ axis})}(x,y,z)\right)^2 \rho(x,y,z)\,dV$$



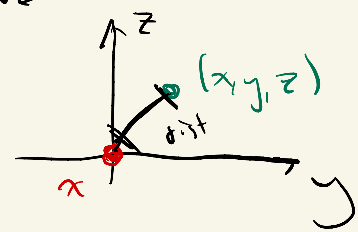
$$= \int_0^1\int_0^1\int_0^1 \left(\sqrt{y^2+z^2}\right)^2 (x+y+z+1)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 (y^2+z^2)(x+y+z+1)\,dx\,dy\,dz$$

$$\vdots$$

$$= \frac{11}{6}$$

By symmetry, $I_x = I_y = I_z = \frac{11}{6}$



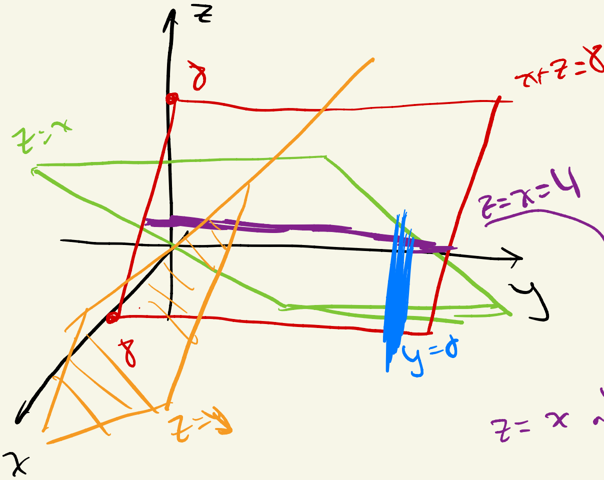
15.5.34

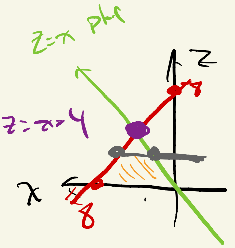
Find the volume bounded by

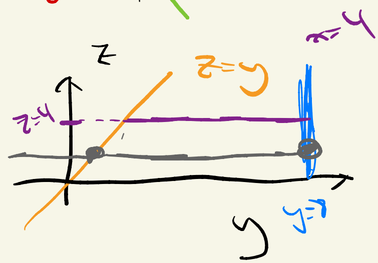
$z = x$ (plane), $x + z = 8$ (plane), $z = y$ (plane), $y = 8$ (plane), $z = 0$

Soln:

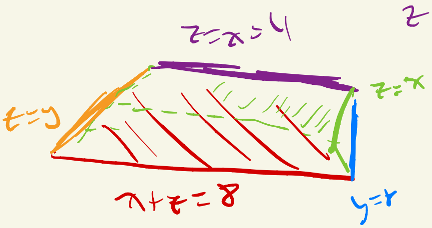
$z = 8 - x$
$x = 8 - z$







$z = x$ and $x + z = 8$
$2x = 8$
$z = x = 4$



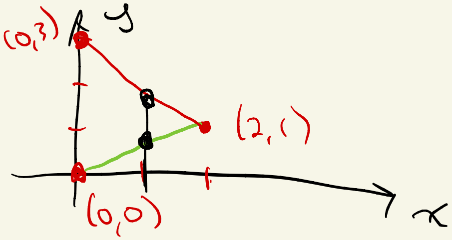
Fix (x, z). How does y vary?

$$\Rightarrow \int_{z=0}^{z=4} \int_{x=z}^{x=8-z} \int_{y=z}^{y=8} dy\, dx\, dz = \frac{320}{3}$$

10:57

Mass of triangular region with vertices $(0,0)$, $(2,1)$, $(0,3)$ and density $\sigma(x,y) = x+y$

Soln:



- $y = 3 - x$
- $y = \frac{1}{2}x$

$$\text{Mass} = \iint_R \sigma(x,y)\, dA$$

$$= \int_{x=0}^{x=2} \int_{y=\frac{1}{2}x}^{y=3-x} x+y \; dy\, dx$$

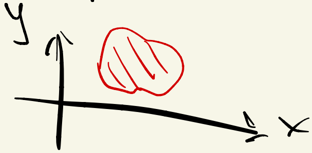
$$= 6$$

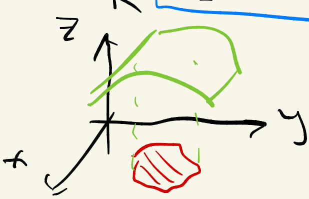
$$\left[z \right]_{z=0}^{z=\sigma(x,y)} = \sigma(x,y)$$

Mass of density function $\sigma(x,y)$ integrated over region $R \subseteq \mathbb{R}^2$

volume under surface $z = \sigma(x,y)$

$$\iint_R \sigma(x,y)\, dA = \iint_R \underbrace{\int_{z=0}^{z=\sigma(x,y)} dz}\; dA$$





Evaluate:

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{6x^2+6y^2}}^{\sqrt{7-x^2-y^2}} 18y \, dz\, dy\, dx$$

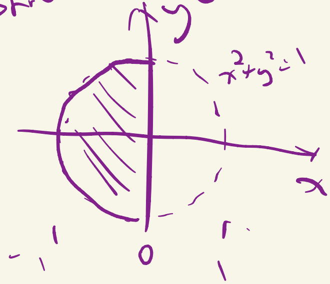
Soln

$$-1 \le x \le 0$$

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$

$$\sqrt{6x^2+6y^2} \le z \le \sqrt{7-x^2-y^2}$$

shadow in xy



top $\quad z = \sqrt{7-x^2-y^2}$

$$z^2 = 7 - x^2 - y^2$$

$$\Rightarrow \quad x^2+y^2+z^2 = 7$$

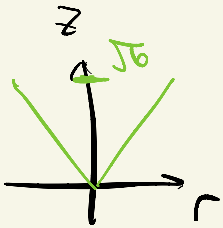
$$\rho^2 = 7$$

sphere of radius $\sqrt{7}$

bottom $\quad z = \sqrt{6x^2+6y^2}$

$$z^2 = 6(x^2+y^2)$$

$$= 6r^2$$



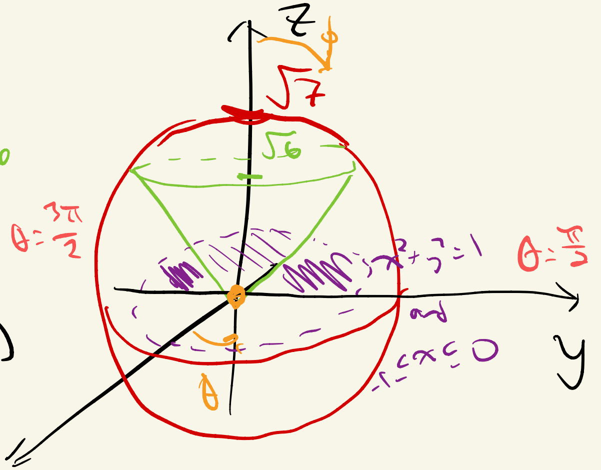
$z = \sqrt{6}|r|$ with $z \ge 0$ (positive branch of $\sqrt{\ }$)

cone with "slope" $\sqrt{6}$

top: sphere of radius $\sqrt{7}$

$$z^2 = 7 - x^2 - y^2$$

bottom: cone with "slope" $\sqrt{6}$



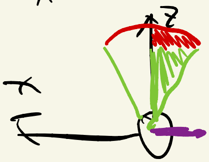
$$z^2 = 6r^2$$

$$\Rightarrow \quad 7 - x^2 - y^2 = 6(x^2+y^2)$$

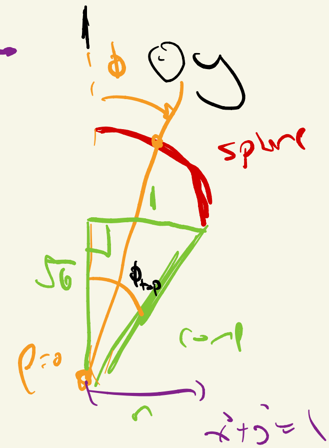
$$7 = 7(x^2+y^2)$$

$$\Rightarrow \quad x^2+y^2 = 1$$

side view



Shadow:



Fix ϕ, θ how does ρ vary?

$$\int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}} \int_{\phi=0}^{\phi=\arctan\left(\frac{1}{\sqrt{6}}\right)} \int_{\rho=0}^{\rho=\sqrt{7}} 18y \; dV$$

$$\Rightarrow \quad \phi_{top} = \arctan\left(\frac{1}{\sqrt{6}}\right) \quad \left(\frac{\text{opposite}}{\text{adjacent}}\right)$$

$$= \boxed{\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{\arctan\left(\frac{1}{\sqrt{6}}\right)} \int_0^{\sqrt{7}} \left(18\rho\sin\phi\sin\theta\right) \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta}$$

$$= 0$$

$$y = r\sin\theta$$
$$= \rho\sin\phi\sin\theta$$